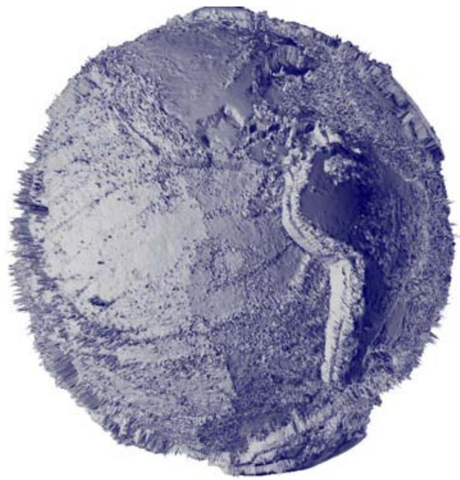
Problematic and Limitations in Modeling Long-Term Lithospheric Deformation.



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CIG, Austin 2006

Outline

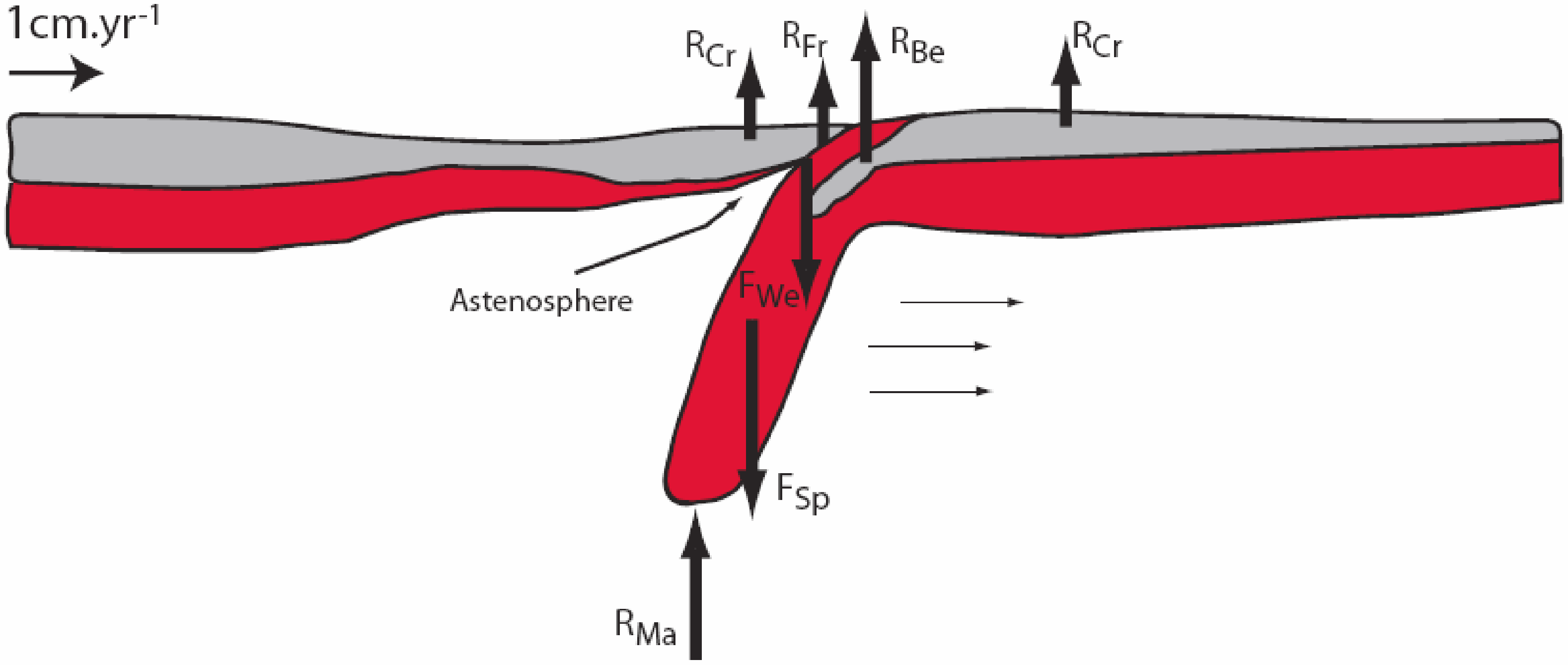
1- Examples of modeling long term lithospheric deformation.
    Subduction, collision, rifting and rollback,etc

2- The constitutive relationship for Earth material.
    Parameterization of the rheology of the lithosphere.

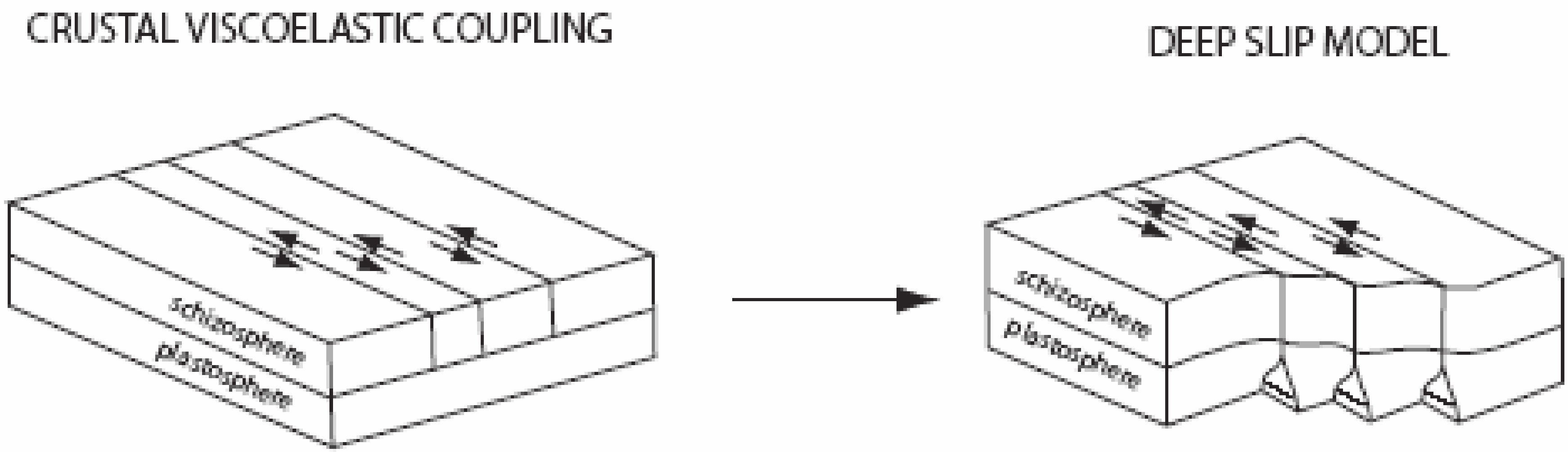
3- What techniques are used and being developed?

4- Two examples of present day techniques.
    SNAC and a Lagrangian FE technique.

Force Balance

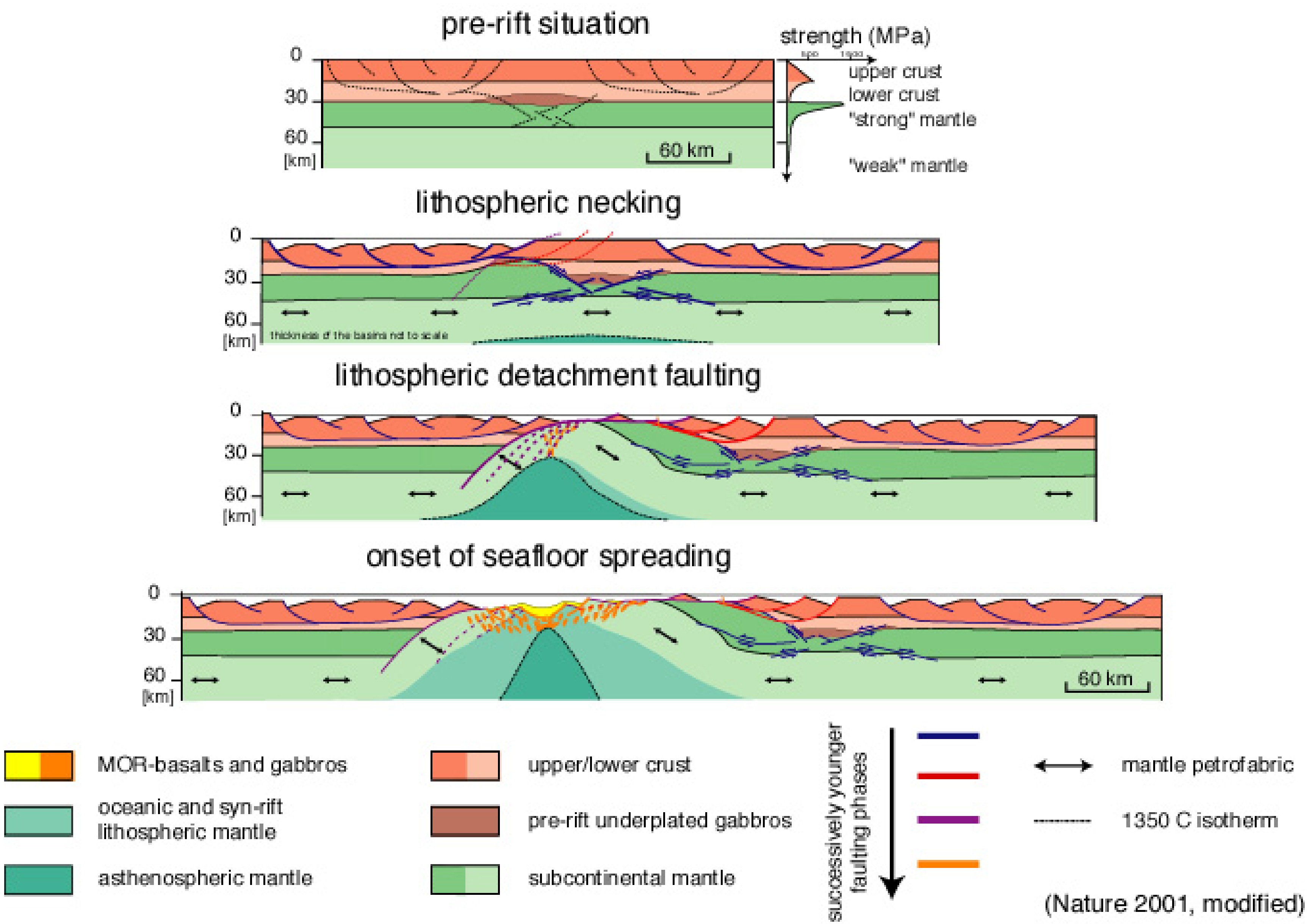


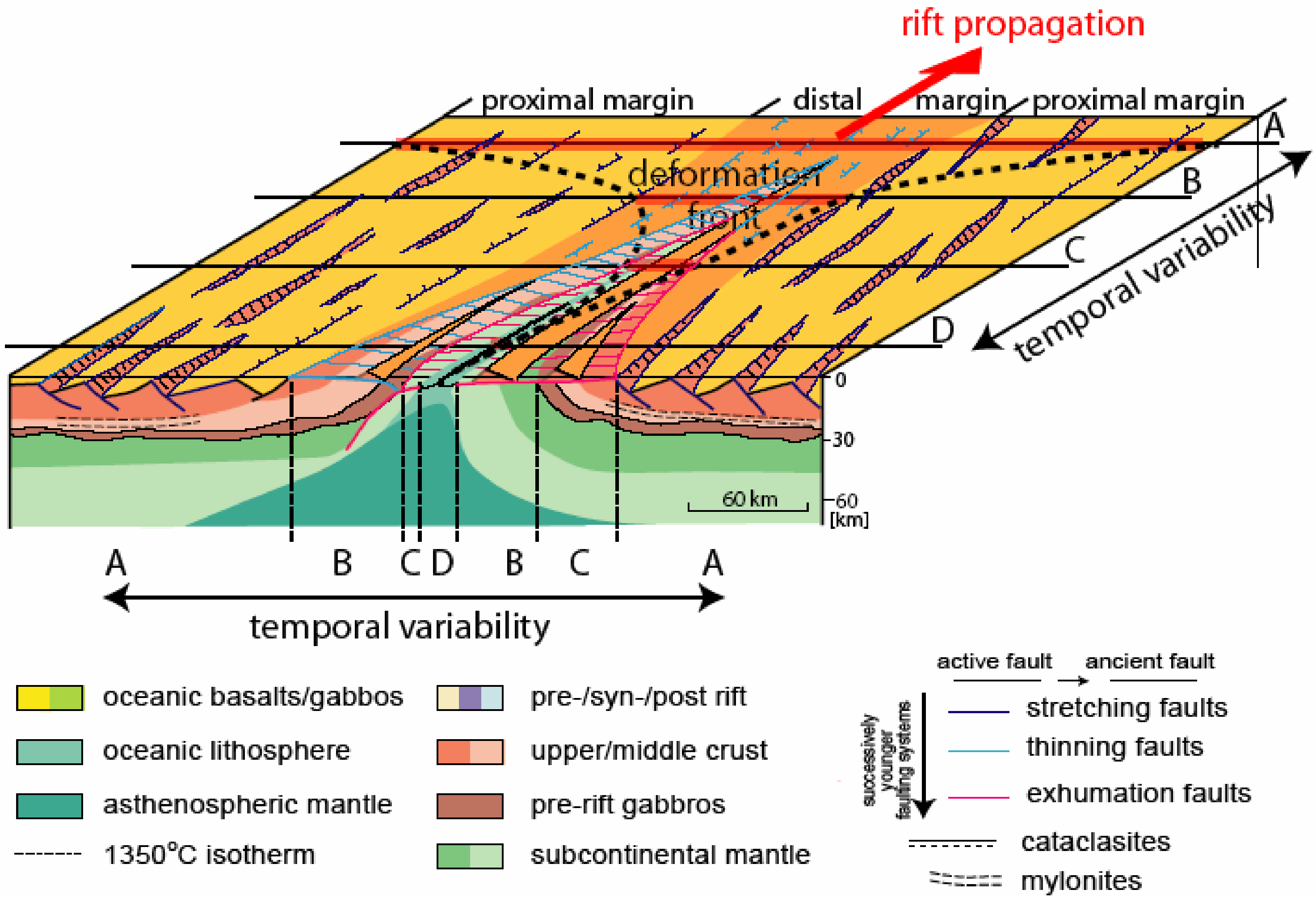
Time dependent behavior.



(Savage et al, 1999) **(Scholz, 2002)**

Rifting of the lithosphere





rift propagation
proximal margin
distal
margin
proximal margin
deformation
front
A
B
C
D
temporal variability
0
30
60
[km]
60 km
A
B
C
D
B
C
A
temporal variability
oceanic basalts/gabbos
oceanic lithosphere
asthenospheric mantle
1350°C isotherm
pre-/syn-/post rift
upper/middle crust
pre-rift gabbros
subcontinental mantle
active fault
ancient fault
successively younger faulting systems
stretching faults
thinning faults
exhumation faults
cataclasites
mylonites

Problems Specific to Lithosphere Geodynamics

- Prediction of structure (Lagrangian Frame)

- Large deformation

- Heterogenous materials

- Non-linear materials

- Free surface - true topography - stress gradients
  moving boundary- erosion and sedimentation.

-Open Boundaries.

-Deformation coupled to temperature and
in the presence of fluids.

Localization (possible use of AMR)

- Localization is assumed to be caused by the weakening of a plastic material as a function of strain, strain rate.

- Theory of localization (Rudnicki and Rice, 1973; Rice, 1976)

  -> "*rate-independent*": pathological mesh-dependence of solution

  -> "*rate-dependent*" : mesh-independent solutions possible. (see Needleman, 1988)

- It can occur associated with various inelastic deformation mechanism such as plasticity and non-linear viscosity

  -> a unified description using "effective stress exponent" (Montesi, 2002).

The problem of deformation

- It can be stated with two equations:

  - A force balance: $$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = 0,$$

  - A constitutive relationship: $$\sigma_{ij} = f(\sigma_{ij}, \varepsilon_{ij}, \dot{\varepsilon}_{ij}, \ldots)$$

  which defines a relationship between the strain and stress.

Constitutive relationship

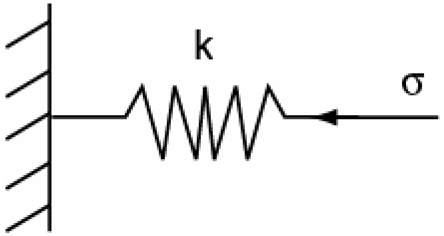
- Geoscientists have access to only few of terms in the parenthesis and they have a hard time to define *f:*

$$\sigma_{ij} = f(\sigma_{ij}, \varepsilon_{ij}, \dot{\varepsilon}_{ij}, T, d, \gamma, \ldots).$$

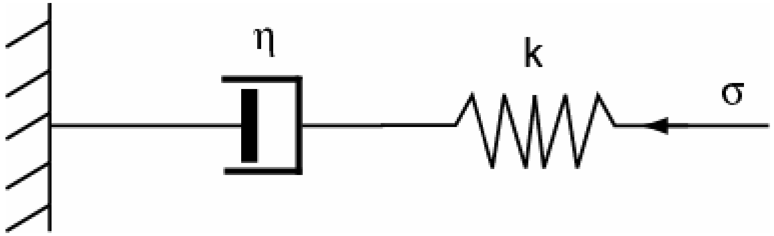
- Geoscientists get strain, strain rates, some temperature, some history…

Constitutive Relationship?

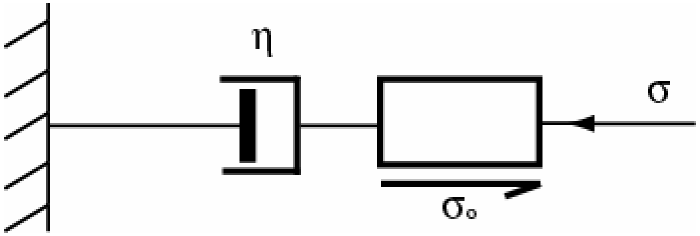
- Elastic



- Maxwell Viscoelastic



- Visco-Plastic



Mohr-Coulomb yield criterion

- Yield potential

$$f(\sigma) = \frac{1}{2}\left(\sigma_3 - \sigma_1\right) + \frac{1}{2}\left(\sigma_3 + \sigma_1\right)\sin\left(\phi\right) - C\cos(\phi)$$

where ϕ is the angle of friction, *C* is cohesive strength, and tensional stresses are positive. Deformation remains elastic for $f(\sigma) < 0$.

Classes of Mechanical Models

Method	Reference Frame	Rheology	Advantages	Disadvantages
EVP – "solid" mechanics based. Finite Element/Finite Difference Methods.	Lagrangian	• Elastic-viscoplastic • Elastic-plastic • Visco-elastic (Maxwell)	• Elastic stress predictor • Associative or non-associative plasticity • Discrete faults can be described by contact elements	• Mesh distortion • Remeshing required • Complex and numerous formulations for finite strain
EVP –Explicit (**FLAC**)		•	• Fast	• Restrictive stability condition
Stokes Flow – Momentumless, fluid dynamics based. Finite Element Method.	Eulerian	• Viscous, non-linear • Rigid-Plastic	• No grid distortion issues • No large strain limitations • Open boundaries (flux)	• Material tracking required by Lagrangian markers (**ALE**, **PIC**) • Difficult to resolve localized deformation (faults) • Isotropic, associative plastic strains only
Particle Methods (Distinct Element Method; Smooth Particle Hydrodynamics) Gridless.	Lagrangian	• Emergent elastic, plastic, viscous	• Accurate representation of discrete strain zones • Variable resolution • No grid distortion issues • Dynamic (momentum)	• Rheology is emergent property • Momentum
Thin Sheets – 2D viscous sheet, solved with FEM	Eulerian or Lagrangian	• Viscous, non-linear	• Reduced Dimensionality by vertical integration of stress and strain • Faults can be included as contact boundaries	• No vertical partitioning or resolution of strain – neglect of components of strain rate

Note: Nothing on numerical methods here

How does it work numerically? www.geodynamics.org

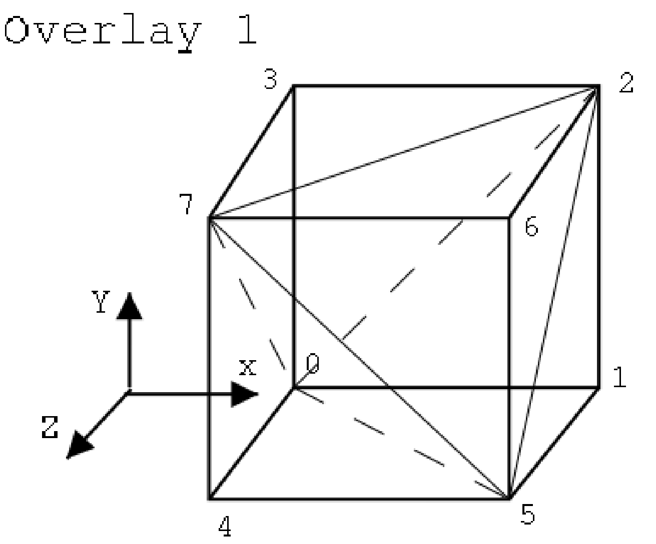
Explicit, implicit, finite difference or finite element approximations.

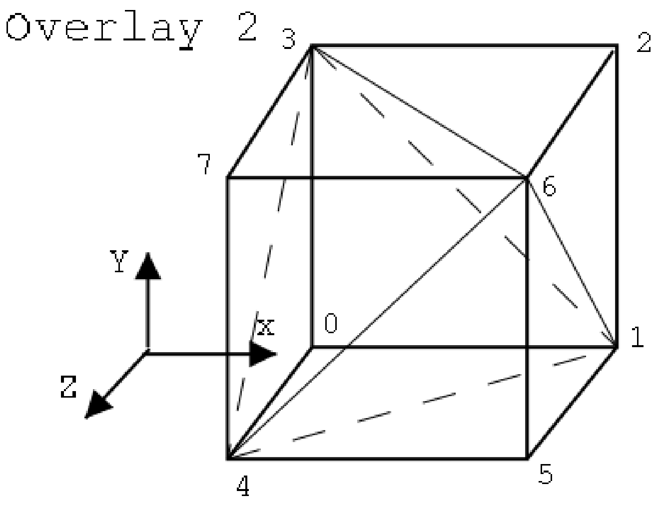
- SNAC (**S**tGermai**N A**nalysis of **C**ontinua).
- GALE (Arbitrary Lagrangian Eulerian)
- Other codes and differences, from CIG (Computational Infrastructure in Geodynamics).

SNAC Grid Discretization

Choi, 2005

- Two overlapped discretization schemes for *mixed discretization*
- Linear tetrahedral element
  - Constant strain-rate within each element
- Zone
  - an 8-node hexahedral element
  - Composed of *two overlays*, each of which is a collection of 5 tetrahedra
  - Symmetric response for symmetric loading





Elements in Finite Difference Method?

Choi, 2005

- SNAC uses "integral definitions" of spatial derivatives
- Assuming a vector quantity is *constant in a tetrahedron*, its spatial derivatives can be approximated by an algebraic expression via Gauss' theorem (Wilkins, 1964)

$$\int_V v_{i,j} dV = \int_S v_i n_j dS$$

$$V v_{i,j} = \sum_{k=1}^{4} \bar{v}_i^k n_j^k S^k$$

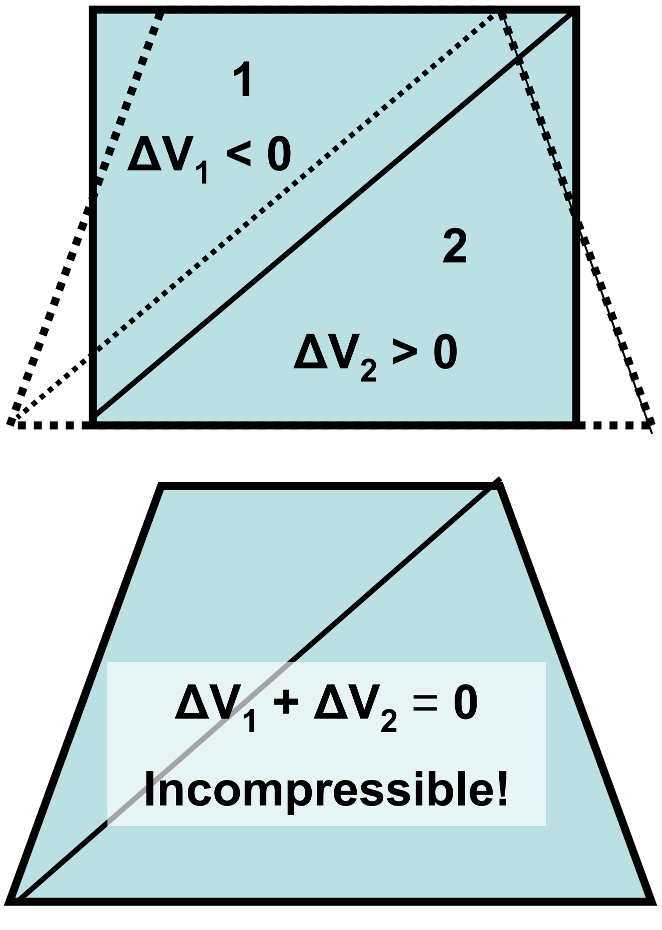
$$v_{i,j} = \frac{1}{V} \sum_{k=1}^{4} \bar{v}_i^k n_j^k S^k$$

- Can express the surface-averaged value as *a sum of nodal values* of an element
- Finally, apply *the principle of virtual work* to get the discrete form of momentum equation
- Looks very similar to FEM

Choi, 2005

Mixed Discretization for incompressibility

< 2-D Analogy >



- Tetrahedral elements cannot deform individually without volume change in particular situations (*e.g.* incompressible plastic flow)
- Substitute the *first invariant* of each tetrahedron with that of a *zone*

SNAC (**S**tGermai**N** **A**nalysis of **C**ontinua)

ADVANTAGES

- Lagrangian.
- Element concept.
- Explicit in constitutive and solution update (easy to implement constitutive update).
- Free surfaces.
- Easy to parallelize.

PROBLEMS

- Very small time step.
- Mesh distortion and remeshing interpolations.
- Inefficient way of dealing with incompressibility.

Finite element Solver with AMR

(W. Bangerth (deal.II), E. Choi)

- The implicit time-discretization of the problem of deformation yields the following system, to be solved on time level *m*:

$$\sigma^m - \Pi(\sigma^{m-1} + \Delta t \, C\varepsilon(v^m)) = 0$$

$$\mathrm{div} \; \sigma^m = F^m$$

- This system is then solved using Newton's method. Since in each Newton step one has to solve a system of the same type as the elastic equations, an elastic solver is the central part of this algorithm.

Forces and Stresses

- *F* are external forces such as gravity, and where the projected stresses are defined by,

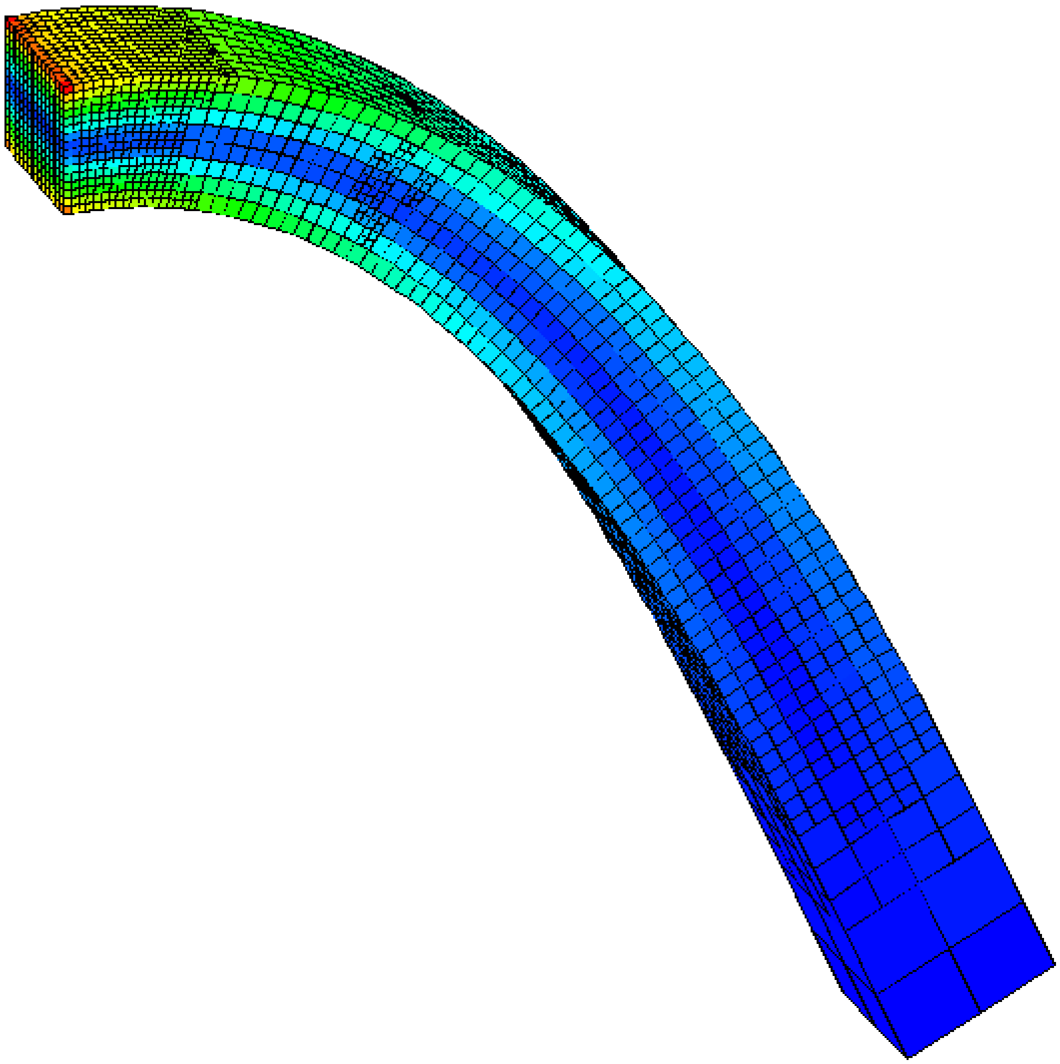
$$\Pi_\sigma(\tau) = \begin{cases} \tau & \text{if } f(\sigma) \le 0 \\ \tau - C\lambda \dfrac{\partial g}{\partial \sigma} & \text{if } f(\sigma) > 0 \end{cases}$$

$f(\sigma)$ is the yield criterion or condition.

Advantages and inconvenient

- It uses stabilized mixed formulation (u-p) finite element.
- Implicit.
- Can deal with incompressibility efficiently.
- With Adaptive mesh refinement (dealii.org)
- It should be faster an more accurate.

- Difficult to implement new rheologies (it may require 2nd order derivative of flow potential)
- Difficult to fully couple to temperature field (shear heating).
- Load balancing is difficult (use PARMETIS).
- Each processors carry the whole mesh.



Arbitrary Lagrangian Eulerian

Ritske Huismans, 2006

- Equilibrium Equation:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = 0$$

- Viscous Incompressible Flow:

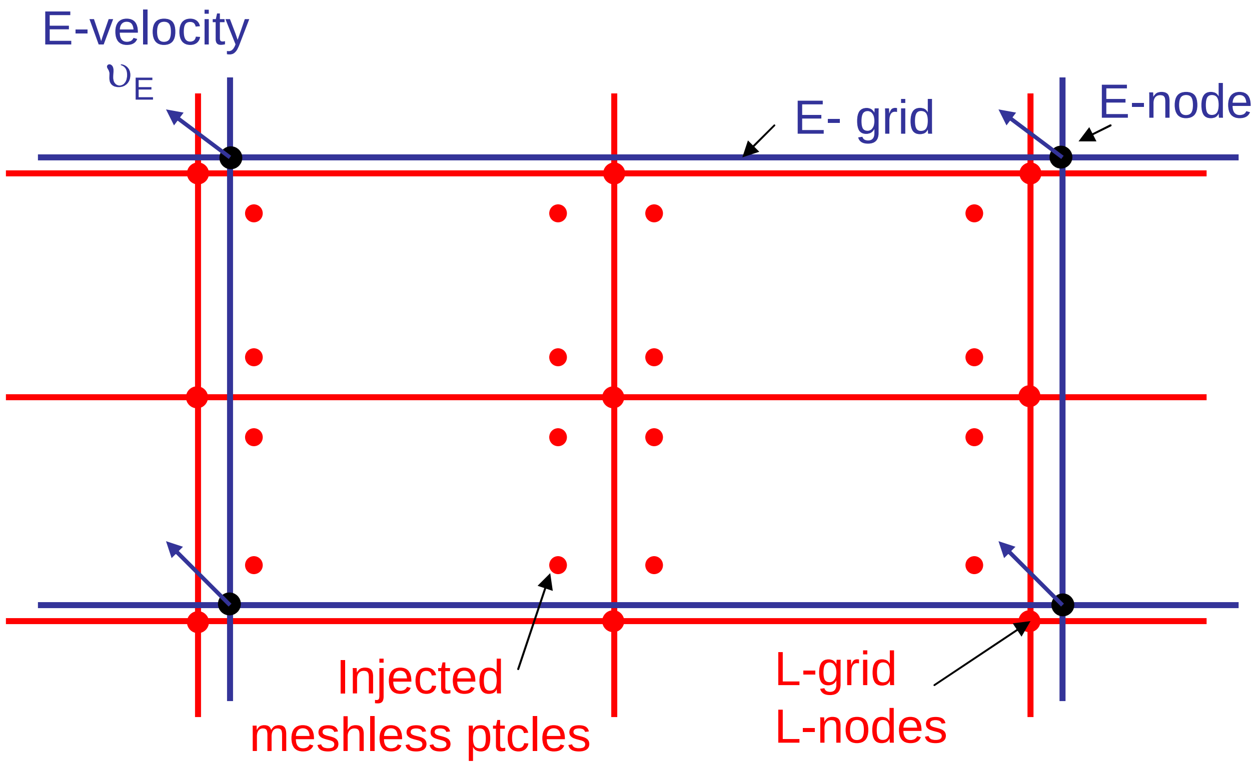
$$\sigma_{ij} = -p\delta_{ij} + 2\eta\dot{\varepsilon}_{ij} \text{ and using } \dot{\varepsilon}_{ij} = \tfrac{1}{2}\left(\partial v_i / \partial x_j + \partial v_j / \partial x_i\right)$$

- Stokes Equation:

$$-\frac{\partial p}{\partial x_j} + \eta \frac{\partial}{\partial x_i}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \rho g_j = 0 \qquad j = 1,2$$

The ALE Techniques: E and L Grids

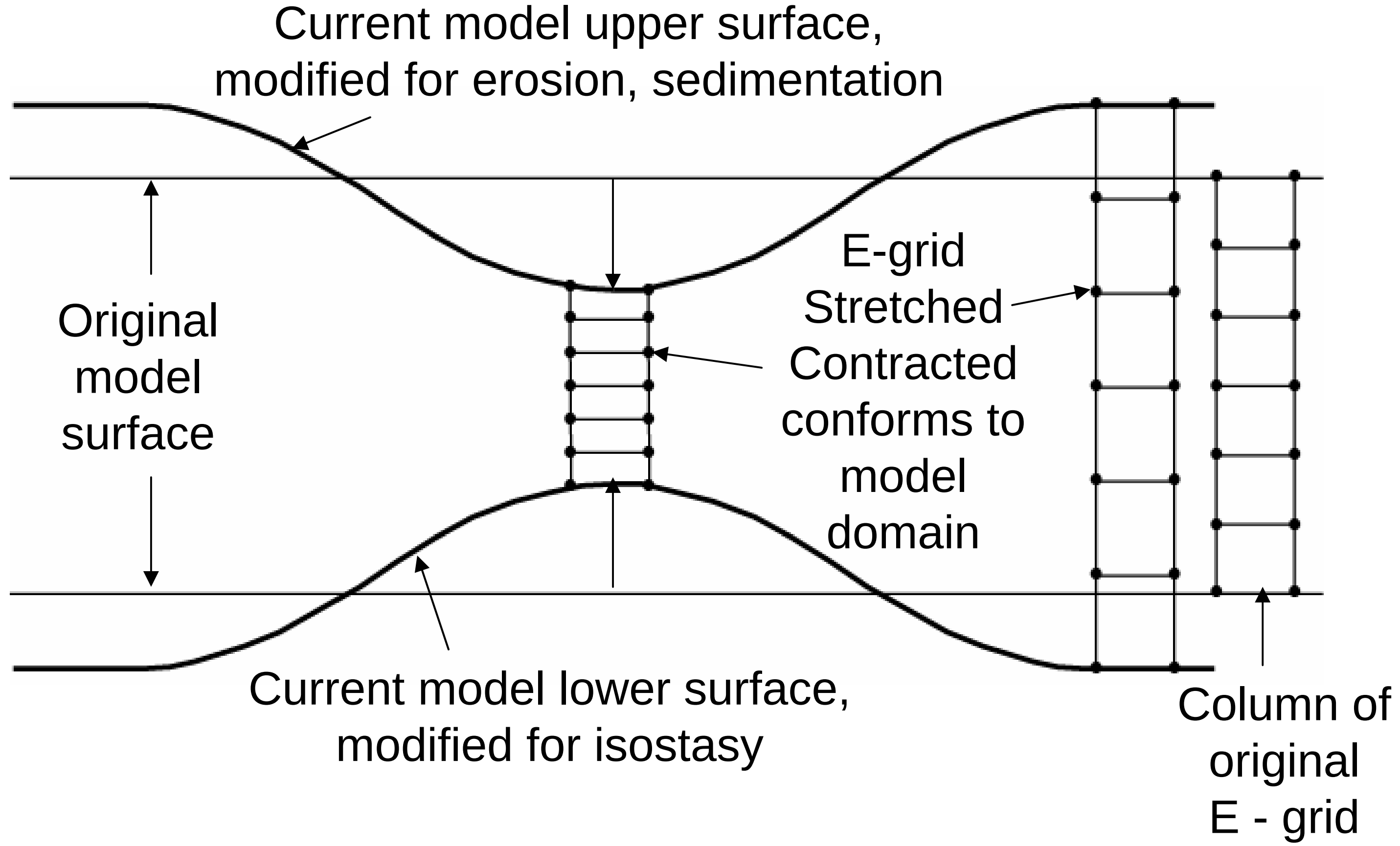
Ritske Huismans, 2006



Finite element problem is solved on the E-grid
L-ptcles are located at the L-grid nodes and are injected within the E-elements
L-ptcles act as a moving ‘cloud’ to advect information on material type, strain, temperature, etc.
This information is re-interpolated back on the E-elements.

E – Remeshing to Conform to Model Domain

Ritske Huismans, 2006



Finite element problem is solved on the E- grid.
The E – grid is stretched/contracted vertically to conform to the material domain.

Advantages and inconvenient

- No mesh distortion.
- Implicit.
- No large strain limitation.
- Open boundaries.

- Limited to isotropic, associative flow laws.
- Particle tracking makes it difficult to resolve localization.

Conclusions

- Patchwork of methods.
- No all encompassing approach.
- Expansive numerical implementations.
- Difficult to think about inversions (for rheology).

Visco-plastic rheology

- “Visco-plastic” is popular
  - Recent studies: Fullsack, 1995; Behn et. Al., 2002; Huismans and Beaumont, 2002, Moresi et al., 2003.
  - Successfully made *localization* happen by applying *frictional strain-softening* to viscous material.

  -> Yielding followed by cohesion / friction angle decrease with accumulating strain.

→ ALE and PIC approaches are possible.

Rheologies

Ritske Huismans, 2006

- Non-Linear Viscous Rheology:

$$\eta_{eff}^{v} = A^{-1/n} \cdot (\dot{I}_2)^{(1-n)/2n} \cdot \exp\left[\frac{Q+Vp}{nRT}\right]$$

- Non-Linear (Frictional) Plastic Rheology

$$(J_2^{/})^{1/2} = c + \alpha \cdot p \cdot Sin\,\phi$$

$$\eta_{eff}^{p} = (J_2^{/})^{1/2} / 2(\dot{I}_2)^{1/2}$$

Viscous-Elastic Flows

Ritske Huismans, 2006

- Compressible Visco-Elastic Flow

$$-\frac{\partial p}{\partial x_i} + \eta_{eff}^{ve}\,\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = -\rho g_j - \frac{\partial}{\partial x_j}\left(\eta\theta\,\tau_{ij}^{n-1}\right)$$

- Effective visco-elastic viscosity

$$\eta_{eff} = \frac{1}{1/(2\mu) + 1/(2\Delta t G)}, \quad \text{and} \quad \theta = \frac{1}{2\Delta t G}$$

- Memory terms:

$$\tau_{ij} = \eta\dot{\varepsilon}_{ij}^{n-1} + \eta\theta\tau_{ij}^{n-1}, \quad \text{and} \quad P = P^{n-1} - \frac{\Delta t K}{3}\frac{\partial v_j^{n-1}}{\partial x_j}$$

Ritske Huismans, 2006

Ritske Huismans, 2006